Vector Fundamentals

An Online Continuing Education Course for Engineers

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Vector Fundamentals

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Introduction

Mechanics is the science of motion and the study of the action of forces on bodies. Mechanics is a physical science incorporating mathematical concepts directly applicable to many fields of engineering such as mechanical, civil, structural and electrical engineering.

Vector analysis is a mathematical tool used in mechanics to explain and predict physical phenomena. The word "vector" comes from the Latin word *vectus* (or *vehere* – meaning to carry). A vector is a depiction or symbol showing movement or a force carried from point A to point B.

A scalar is a quantity, like mass (14 kg), temperature (25°C), or electric field intensity (40 N/C) that only has magnitude and no direction. On the other hand, a vector has both magnitude and direction. Physical quantities that have magnitude and direction can be represented by the length and direction of an arrow. The typical notation for a vector is as follows:

$$\vec{\mathbf{A}} \text{ or simply } \mathbf{A}$$

Note: vectors in this course will be denoted as a boldface letter: **A**.

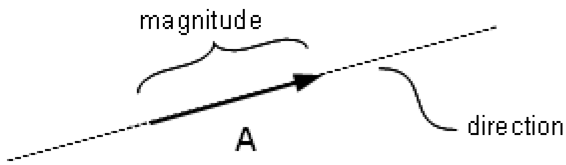

Figure 1 – Illustration of a vector

Vectors play an important role in physics (specifically in kinematics) when discussing velocity and acceleration. A velocity vector contains a scalar (speed) and a given direction. Acceleration, also a vector, is the rate of change of velocity.

Vector Decomposition

Cartesian Coordinate System

Consider a 3-dimensional Cartesian coordinate system:

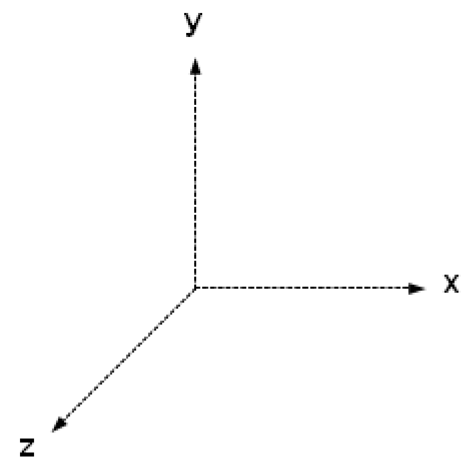

Figure 2 - Cartesian (or rectangular coordinate system)

A Cartesian (or rectangular) coordinate system has three mutually perpendicular axes: x, y and z. A vector in this coordinate system will have components along each axis.

A unit vector is a vector along an axis (x, y or z) with a length of one. Let the unit vector along the x-axis be **i** and the unit vector along the y-axis be **j** and the unit vector along the z-axis be **k**. The Cartesian coordinate system with three unit vectors is shown in Figure 3.

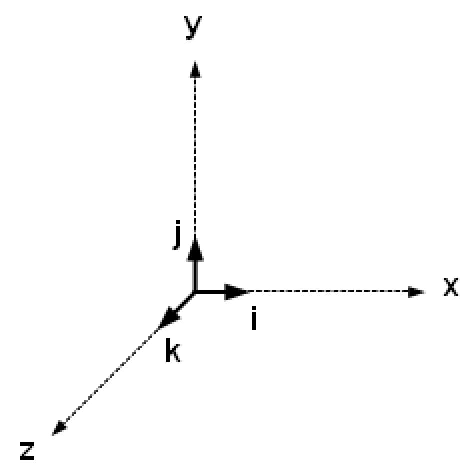

Figure 3 - Rectangular coordinate system showing unit vectors

A vector can connect two points in space as in Figure 4.

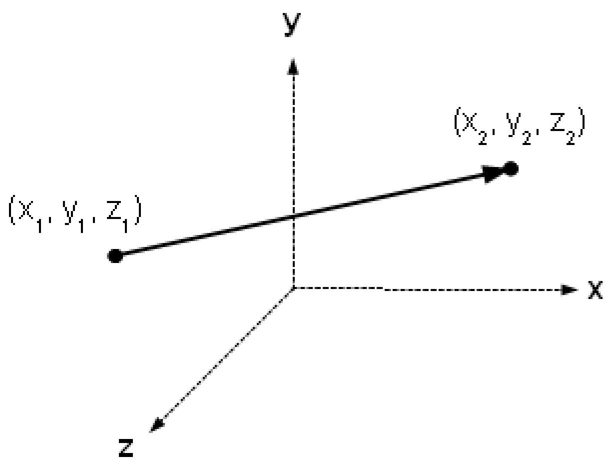

Figure 4 - Vector connecting two points in space

Components of a Vector

In a Cartesian coordinate system, the components of a vector are the projections of the vector along the x, y and z axes. Consider the vector **A**. The vector **A** can be broken down into its components along each axis: A_x, A_y and A_z in the following manner:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

Note that the vectors **i**, **j** and **k** are the unit vectors along each corresponding axis. The unit vectors **i**, **j** and **k** each have a length of one, and the magnitudes along each direction are given by A_x, A_y and A_z.

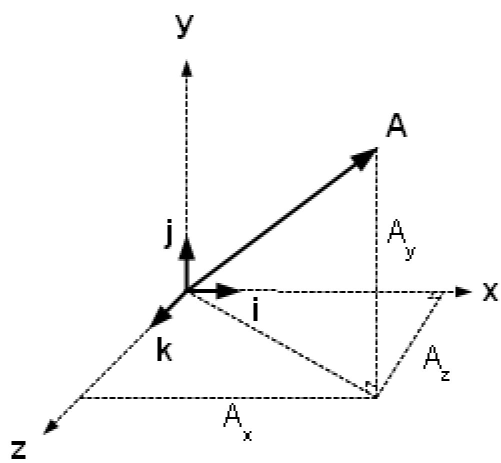

Figure 5 - Vector decomposition showing components along each axis

Trigonometry is utilized to compute the vector components A_x, A_y and A_z. Consider a vector in 2-dimensional space:

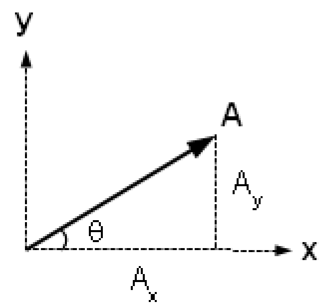

Figure 6 - Vector in 2-dimensional space

The components of this 2-dimensional vector are computed with respect to the angle θ as follows:

$A_x = A\cos\theta$

and

$A_y = A\sin\theta$

Where A is the magnitude of **A** given by $A = \sqrt{A_x^2 + A_y^2}$

For example, let

A = 5 and θ = 36.8°

then

$A_x = 5\cos(36.8°)$
$= 5(0.8)$
$= 4$

and

$A_y = 5\sin(36.8°)$
$= 5(0.6)$
$= 3$

Therefore, the vector (in rectangular form) is: $\mathbf{A} = 4\mathbf{i} + 3\mathbf{j}$

As a result of the Pythagorean Theorem from trigonometry the magnitude of a vector may be calculated by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The magnitude may also be denoted as: $A = |\mathbf{A}|$ or $A = \|\mathbf{A}\|$

The magnitude of a vector is the length of the vector. It is a scalar (length only) with no direction. In physics, for example, speed is a scalar and velocity is a vector, so speed is the magnitude of the velocity vector.

Now consider, once again, the vector in 3-dimensional space:

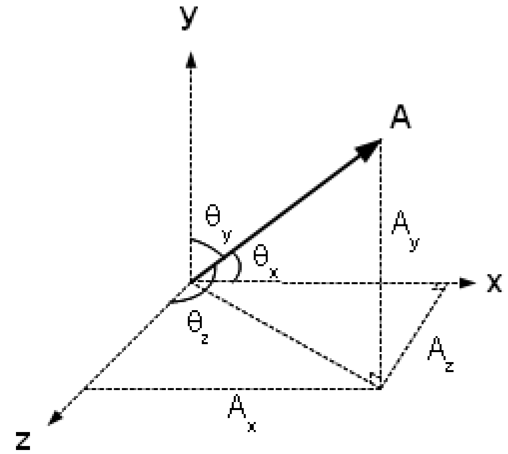

Figure 7 - Vector decomposition showing angles to axes

The components of this 3-dimensional vector are computed with respect to the angles θ_x, θ_y and θ_z as follows:

$$A_x = A\cos\theta_x$$
$$A_y = A\cos\theta_y$$
$$A_z = A\cos\theta_z$$

where $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ is the magnitude of **A** and $\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$.

A unit vector can be constructed along a vector by dividing the vector by its magnitude. The result is a vector along the same direction as the original vector with magnitude 1. Consider the unit vector **a**:

$$\mathbf{a} = \frac{\mathbf{A}}{A}$$

$$\mathbf{a} = \frac{A_x\mathbf{i}}{A} + \frac{A_y\mathbf{j}}{A} + \frac{A_z\mathbf{k}}{A}$$

This is also called vector normalization. The magnitude of **a** is one.

Properties of a Vector

Addition

Vector addition is accomplished by adding the components (A_x, A_y and A_z) of one vector to the associated components (B_x, B_y and B_z) of another vector:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

For example, consider the following vectors **A** and **B**:

$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$
$\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

then

$\mathbf{A} + \mathbf{B} = (1 + 3)\mathbf{i} + (2 + 1)\mathbf{j} + (5 + 2)\mathbf{k}$
$= 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$

Commutative Property
Vector addition follows the commutative property:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative Property
Vector addition also follows the associative property:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

Scalar Multiplication
Vectors can be multiplied by real numbers (called scalars). To accomplish this the vector components (A_x, A_y and A_z) are each multiplied by the real number (n):

$$n\mathbf{A} = nA_x\mathbf{i} + nA_y\mathbf{j} + nA_z\mathbf{k}$$

For example, let n = 5, and

the

Con
Scal ive property:

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Scalar roperty: